**The Trapezoidal Rule**

We can also approximate the value of a definite integral by using trapezoids rather than rectangles. In Figure 2.5.2 , the area beneath the curve is approximated by trapezoids rather than by rectangles.

The trapezoidal rule for estimating definite integrals uses trapezoids rather than rectangles to approximate the area under a curve. To gain insight into the final form of the rule, consider the trapezoids shown in Figure 2.5.2 . We assume that the length of each subinterval is given by Δx . First, recall that the area of a trapezoid with a height of h and bases of length b1 and b2 is given by Area=1/2h(b1+b2) . We see that the first trapezoid has a height Δx and parallel bases of length f(x0) and f(x1) . Thus, the area of the first trapezoid in Figure 2.5.2 is

1/2Δx(f(x0)+f(x1)).

The areas of the remaining three trapezoids are

1/2Δx(f(x1)+f(x2)),1/2Δx(f(x2)+f(x3)), and 1/2Δx(f(x3)+f(x4)).

Consequently,

∫baf(x)dx≈1/2Δx(f(x0)+f(x1))+1/2Δx(f(x1)+f(x2))+1/2Δx(f(x2)+f(x3))+1/2Δx(f(x3)+f(x4)).

After taking out a common factor of 1/2Δx and combining like terms, we have

∫baf(x)dx≈Δx/2(f(x0)+2f(x1)+2f(x2)+2f(x3)+f(x4)).

Generalizing, we formally state the following rule.

**Error Bounds on the Trapezoidal Rules**

~~In the two previous examples, we were able to compare our estimate of an integral with the actual value of the integral; however, we do not typically have this luxury. In general, if we are approximating an integral, we are doing so because we cannot compute the exact value of the integral itself easily. Therefore, it is often helpful to be able to determine an upper bound for the error in an approximation of an integral. The following theorem provides error bounds for the midpoint and trapezoidal rules. The theorem is stated without proof.~~

Let f(x) be a continuous function over [a,b] , having a second derivative f′′(x) over this interval. If M is the maximum value of |f′′(x)| over [a,b] , then the upper bounds for the error in using Tn to estimate ∫baf(x)dx are

Error inTn≤M(b−a)312n2(2.5.5)

.

We can use these bounds to determine the value of n necessary to guarantee that the error in an estimate is less than a specified value.

**Simpson’s Rule**

With the midpoint rule, we estimated areas of regions under curves by using rectangles. In a sense, we approximated the curve with piecewise constant functions. With the trapezoidal rule, we approximated the curve by using piecewise linear functions. What if we were, instead, to approximate a curve using piecewise quadratic functions? With Simpson’s rule, we do just this. We partition the interval into an even number of subintervals, each of equal width. Over the first pair of subintervals we approximate ∫x2x0 f(x)dx with ∫x2 x0 p(x)dx , where p(x)=Ax2+Bx+C is the quadratic function passing through (x0,f(x0)),(x1,f(x1)), and (x2,f(x2)) (Figure 2.5.4 ). Over the next pair of subintervals we approximate ∫x4x2f(x)dx with the integral of another quadratic function passing through (x2,f(x2)),(x3,f(x3)), and (x4,f(x4)). This process is continued with each successive pair of subintervals.

To understand the formula that we obtain for Simpson’s rule, we begin by deriving a formula for this approximation over the first two subintervals. As we go through the derivation, we need to keep in mind the following relationships:

f(x0)=p(x0)=Ax20+Bx0+C

f(x1)=p(x1)=Ax21+Bx1+C

f(x2)=p(x2)=Ax22+Bx2+C

x2−x0=2Δx , where Δx is the length of a subinterval.

x2+x0=2x1, since x1=(x2+x0)2 .

Thus,

∫x2x0f(x)dx≈∫x2x0p(x)dx

=∫x2x0(Ax2+Bx+C)dx

=(A3x3+B2x2+Cx)∣∣∣x2x0 Find the antiderivative.

=A3(x32−x30)+B2(x22−x20)+C(x2−x0) Evaluate the antiderivative.

=A3(x2−x0)(x22+x2x0+x20)+B2(x2−x0)(x2+x0)+C(x2−x0)

=x2−x06(2A(x22+x2x0+x20)+3B(x2+x0)+6C) .Factor outx2−x06

=Δx3((Ax22+Bx2+C)+(Ax20+Bx0+C)+A(x22+2x2x0+x20)+2B(x2+x0)+4C) Rearrange the terms. Note:Δx=x2−x02

=Δx3(f(x2)+f(x0)+A(x2+x0)2+2B(x2+x0)+4C) Factor and substitute:f(x2)=Ax22+Bx2+Candf(x0)=Ax20+Bx0+C.

=Δx3(f(x2)+f(x0)+A(2x1)2+2B(2x1)+4C) Substitutex2+x0=2x1.Note:x1=x2+x02,the midpoint.

=Δx3(f(x2)+4f(x1)+f(x0)). Expand and substitutef(x1)=Ax21+Bx1+C.

If we approximate ∫x4x2f(x)dx using the same method, we see that we have

∫x4x0f(x)dx≈Δx3(f(x4)+4f(x3)+f(x2)).

Combining these two approximations, we get

∫x4x0f(x)dx=Δx3(f(x0)+4f(x1)+2f(x2)+4f(x3)+f(x4)).

The pattern continues as we add pairs of subintervals to our approximation. The general rule may be stated as follows.

Simpson’s Rule

Assume that f(x) is continuous over [a,b] . Let n be a positive even integer and Δx=b−an . Let [a,b] be divided into n subintervals, each of length Δx , with endpoints at P={x0,x1,x2,…,xn}. Set

Sn=Δx3(f(x0)+4f(x1)+2f(x2)+4f(x3)+2f(x4)+⋯+2f(xn−2)+4f(xn−1)+f(xn)).(2.5.9)

Then,

limn→+∞Sn=∫baf(x)dx.

Just as the trapezoidal rule is the average of the left-hand and right-hand rules for estimating definite integrals, Simpson’s rule may be obtained from the midpoint and trapezoidal rules by using a weighted average. It can be shown that S2n=(23)Mn+(13)Tn .

**Error Bound for Simpson’s Rule**

Let f(x) be a continuous function over [a,b] having a fourth derivative, f(4)(x) , over this interval. If M is the maximum value of ∣∣f(4)(x)∣∣ over [a,b] , then the upper bound for the error in using Sn to estimate ∫baf(x)dx is given by

Error inSn≤M(b−a)5180n4.(2.5.10)

**Absolute and Relative Error**

An important aspect of using these numerical approximation rules consists of calculating the error in using them for estimating the value of a definite integral. We first need to define absolute error and relative error.

Definition: absolute and relative error

If B is our estimate of some quantity having an actual value of A , then the absolute error is given by |A−B| .

The relative error is the error as a percentage of the actual value and is given by

∣∣∣A−B/A∣∣∣⋅100%.